Parrondo's games and the zipping algorithm

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ABSTRACT

We study the relation between the discrete–time version of the flashing ratchet known as Parrondo's games and a compression technique used very recently with thermal ratchets for evaluating the transfer of information – negentropy – between the Brownian particle and the source of fluctuations. We present some results concerning different versions of Parrondo's games showing, for each case, a good qualitative agreement between the gain and the inverse of the entropy.

Keywords: Ratchets, Parrondo's paradox, information theory

1. INTRODUCTION

In recent years, the microscopic field of Brownian particles has received much attention due to new transport phenomena under certain conditions. The main features involve breaking of spatial inversion symmetry and a thermal bath that drives the system out of equilibrium[1] in order to obtain directed transport. The former condition is accomplished through an asymmetric potential, usually a ratchet–like potential. The latter can be achieved in different ways, either by a periodic or stochastic forcing – what it is known as *pulsating ratchets*, or by an additive driving force unbiased on average – known as *tilting ratchets*. The *flashing ratchet* model corresponds to the class of pulsating ratchets and it consists on switching on and off either periodically or stochastically a ratchet potential. This model has been used recently for DNA transport[2] and separation of biological macromolecules.[3]

Recently, Arizmendi *et. al*[4] have quantified the transfer of information – negentropy – between the Brownian particle and the nonequilibrium source of fluctuations acting on it. They coded the particle motion into a binary string of 0's and 1's, and then zipped the resulting binary file using the Lempel and Ziv algorithm[5] – available through the `gzip` program. In this way they obtained an estimation of the entropy per character h, as the ratio between the length of the zipped file and the original length file, when the text length tends to infinity. They used this method to estimate the entropy per character of the ergodic source for different values of the flipping rate, with the result that there exists a close relation between the current in the ratchet and the net transfer of information in the system.

The aim of this paper is to apply this technique to a discrete–time and space version of the Brownian ratchet, known in the literature as Parrondo's paradox.

Parrondo's paradox[6–12] is based on a combination of two negatively biased games – losing games – which when combined give rise to a positively biased game – a winning game. This paradox appeared as a translation of the physical model of the Brownian ratchet into game–theoretic terms. But although it was inspired by the flashing ratchet model, there has been no quantitative demonstration of their relation until very recently.[13–15]

More precisely, the so called Parrondo's paradox is based on the combination of two games (Parrondo's games). One of them, say game A, is a simple coin tossing game where the player has a probability p of winning one unit of capital and a probability $1 - p$ of losing one unit of capital. The second game, game B, is a capital dependent game, where the probability of winning depends on the capital of the player modulo a given number

M. For the original games, M is set to three and there are only two possible values for the winning probabilities, p_0 when the capital is a multiple of three and p_1 otherwise. Their numerical values are

$$\begin{cases} p = & \frac{1}{2} - \epsilon, \\ p_0 = & \frac{1}{10} - \epsilon, \\ p_1 = & \frac{3}{4} - \epsilon, \end{cases} \qquad (1)$$

here ϵ is nothing but a biasing parameter that converts games A and B into losing games. When $\epsilon = 0$ it can be demonstrated that the fairness condition is fulfilled for both games, that is $\prod_{j=0}^{M-1} p_i = \prod_{j=0}^{M-1}(1 - p_i)$. As soon as $\epsilon > 0$ the latter condition no longer applies and A and B are both losing games.

The combination game, game AB, is obtained mixing game A and game B with probability γ. As a result of this combination we obtain a winning game, even when ϵ is equal to zero. The corresponding winning probabilities are given by the following expressions

$$\begin{cases} p_0' = & \gamma p + (1 - \gamma)p_0, \\ p_1' = & \gamma p + (1 - \gamma)p_1. \end{cases} \qquad (2)$$

Several other versions of the games have been introduced. In the so-called cooperative games,[16, 17] one considers an ensemble of interacting players. In the history dependent games,[18, 19] the probabilities of winning depend on the history of previous results of winnings and losing. Finally, in the games with self–transition,[20] there is a non–zero probability that the capital can stay the same (not winning or losing) in a given toss of the coins.

Some previous works in the literature has related Parrondo's games and information theory. Pearce, in Ref.,[21] considers the relation between the entropy and the fairness of the games, and the region of the parameter space where the entropy of game A is greater than that of B and AB. Harmer *et. al*[22] studied the relation between the fairness of games A and B and the entropy rates considering two approaches. The first one consisted on calculating the entropy rates not taking into account the correlations present on game B, finding a good agreement between the region of maximum entropy rates and the region of fairness. In the second approach they introduced these correlations obtaining lower entropy rates and no significant relation between fairness and entropy rates for game B.

In this paper we aim to relate the current or gain in Parrondo's games with the variation of information entropy of the binary file generated. Following the same approach as in Ref.,[4] we have carried out simulations for different versions of Parrondo's games, including the original version, the history dependent games, Parrondo's games with self–transition and cooperative games mentioned above.

2. SIMULATION RESULTS

We have carried out numerical simulations of the different versions of the games. In the simulations, the evolution of the capital of the player has been converted to a binary string of 0's and 1's, corresponding to a decrease and an increase of capital respectively. We must remark that each one of these characters is stored as a byte (eight bits). A binary file has then been created for different values of the γ parameter. In order to obtain the desired effect there must be a certain delay time, which we will denote by δ_t, between measurements of the capital of the player.

The binary files created have been zipped using the `gzip` (v. 1.3) program, to obtain an estimation of the entropy per character h. For a better comparison with the current – or gain – generated when alternating between games A and B, we will define $R = \frac{1}{h}$, accounting, in some sense, for the known information about the system. In the Appendix there is an extract of the C program used for the case of the original Parrondo's games.

In Fig. 1 we compare the gain when alternating between games A and B with probability γ with the value of R obtained for different values of the δ_t parameter. The numerical curves are shifted to the origin, that is, we plot $\Delta R = R_\gamma - R_{\gamma=0}$.

We find a good qualitative agreement between the increase in the gain and the decrease (increase) in entropy (R) as the γ parameter is varied. Although both curves do not fully coincide, the important fact to consider here
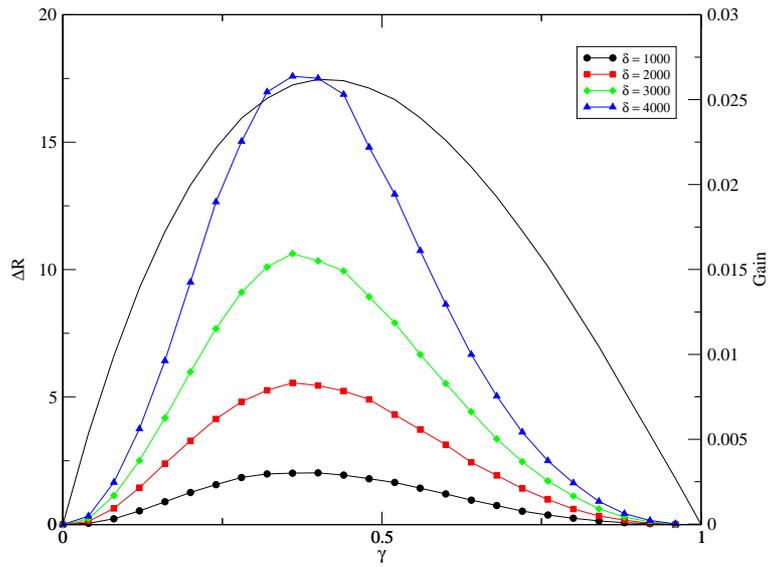
Figure 1. Comparison of the gain in the original Parrondo's games with ΔR calculated for different δ_t and γ values. The probabilities for the Parrondo game are $p = \frac{1}{2}$, $p_0 = \frac{1}{10}$ and $p_1 = \frac{3}{4}$. The values used are $\delta_t = 1000$ (circles), 2000 (squares), 3000 (diamonds) and 4000 (triangles).
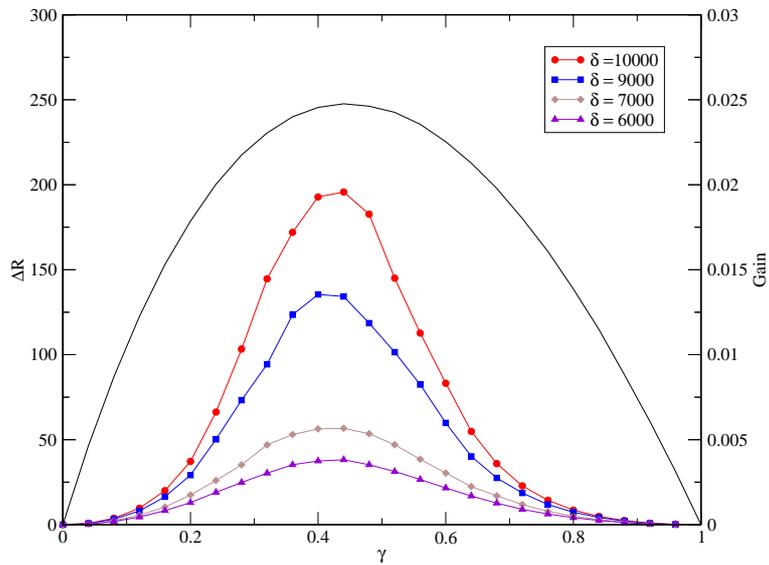
Figure 2. Comparison between the inverse of the entropy per character R, calculated for different δ_t values, and the gain for the Parrondo's games A and B with self–transition varying γ. The values for the probabilities are: $p = \frac{9}{20}$, $r = \frac{1}{10}$, $p_0 = \frac{3}{25}$, $r_0 = \frac{2}{5}$, $p_1 = \frac{3}{5}$ and $r_1 = \frac{1}{10}$. See Ref.[20] for an explanation of these parameters. The values used are $\delta_t = 6000$ (triangles), 7000 (diamonds), 9000 (squares) and 10000 (circles).
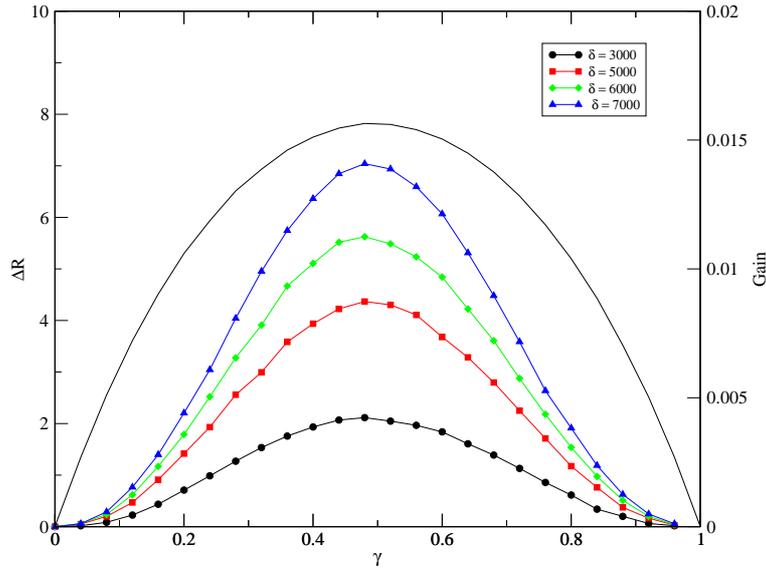
Figure 3. Curve for the inverse of the entropy per character R and the gain for the history dependent games, alternating between two games with probabilities: $p_1 = \frac{9}{10}$, $p_2 = p_3 = \frac{1}{4}$, $p_4 = \frac{7}{10}$; $q_1 = \frac{2}{5}$, $q_2 = q_3 = \frac{3}{5}$ and $q_4 = \frac{2}{5}$. See Ref.[18] for an explanation of these parameters. The values for the δ_t are: 3000 (circles), 5000 (squares), 6000 (diamonds) and 7000 (triangles).

is that the γ value for which there is the maximum decrease in entropy agrees with the value for the maximum gain in the games. This decrease in the entropy of the system implies that there exists an increase in the amount of information that we know about the system.

In Fig. 2 we compare the gain curve and the entropy curve for the case of Parrondo's games with self–transition. Again in this case the maximum gain coincides with the γ value for the minimum entropy per character for all values of δ_t.

Finally, in Figs. 3 and 4 we show the comparison in the case of the history dependent games, and cooperative games, sharing all them the same features as the previous games analyzed before.

3. THEORETICAL ANALYSIS

In this section we develop a simple argument that explains the observed relation between the gain of the games and their entropy. In the seminal work by Shannon[23] the entropy per character of a text produced by an ergodic source is given by the following expression

$$H = -\sum_i p_i \cdot \log(p_i) \tag{3}$$

where p_i denotes the probability that the source will 'emit' a given symbol a_i, and the sum is taken over all possible symbols that the source can emit.

If the source can be found in several states, each one with probability P_j, then the entropy reads

$$H = \sum_{j=1}^{M} P_j H_j = -\sum_{j,i} P_j \, p_i^j \, \log(p_i^j). \tag{4}$$

In the last equation p_i^j denotes the probability of emitting the symbol a_i when the source is found in the state j.
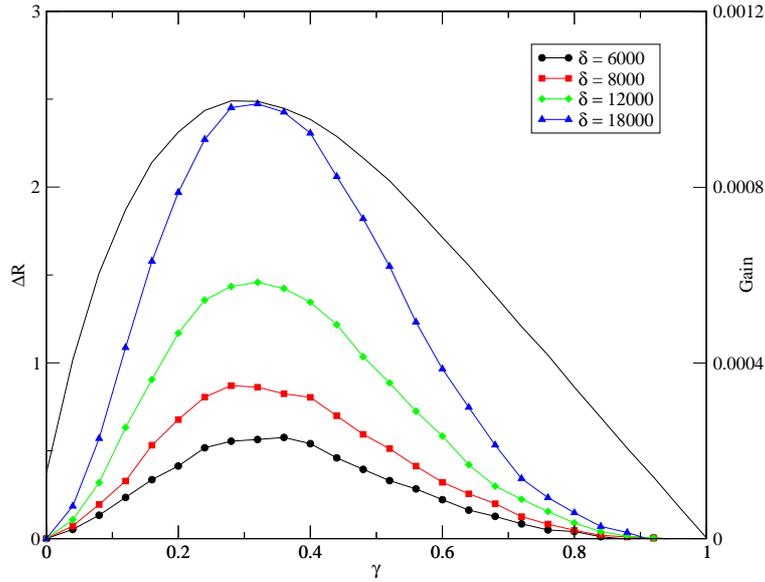
Figure 4. Curve for the inverse of the entropy per character R and the gain for the cooperative version with probabilities: $p = \frac{1}{2}$, $p_1 = 1$, $p_2 = p_3 = \frac{16}{100}$, $p_4 = \frac{7}{10}$ and $N = 150$ players for different values of the γ parameter. See Ref.[16] for an explanation of these parameters. The values of the δ_t are: 6000 (circles), 8000 (squares), 12000 (diamonds) and 18000 (triangles).

Remember that we measure and store the capital of the player after δ_t rounds have been played. When δ_t equals one it means that we measure the capital of the player *in each* round. Translating this into entropic terms we have that, considering the original Parrondo's games, our source will be in two possible states, say, when the capital is multiple of three and otherwise. From Markov chain theory[24] we know that the stationary probabilities of finding the capital in each of the states are given by

$$\Pi_0 = \frac{1}{D}(1 - p_1 + p_1^2)$$

$$\Pi_1 = 1 - \Pi_0 = \frac{1}{D}(2 - p_0 - p_1 + 2\,p_0 p_1)$$

(5)

where $D = 3 - p_0 - 2p_1 + 2p_0p_1 + p_1^2$.

It is interesting to consider our source (the capital of the player) as composed of several sources, each one with an homogeneous statistical structure, i.e. they are ergodic sources. Each source then will have a determined probability of 'emitting' a zero or a one, as the player has a different winning probability depending on whether his capital is multiple of three or not. If we consider that the source will emit the symbol 1 with probability p_i and 0 with probability $1 - p_i$ then Eq. 4 reads

$$
\begin{aligned}
H &= \sum_{j=1}^{M} \Pi_j H_j = -\sum_{j,i} \Pi_j p_i^j \log(p_i^j) \\
&= -\Pi_0 \cdot (p_1 \cdot \log(p_1) + (1 - p_1) \cdot \log(1 - p_1)) - (1 - \Pi_0) \cdot (p_2 \cdot \log(p_2) + (1 - p_2) \cdot \log(1 - p_2))
\end{aligned}
$$

(6)

In Fig. 5 we plot the inverse of Eq. 6 for the values of p_i^j and Π_i corresponding to the original Parrondo's games together with the numerical curve obtained when zipping the files obtained with $\delta_t = 1$. It is worth noting how the negentropy decreases as γ increases, translating into an increase of known information about the system.

Alternatively, we can now estimate how the entropy changes for $\delta_t \gg 1$, when we allow a large amount of time between measures. As δ_t is increased, the statistical description used above cannot be used anymore. For higher δ_t values, the system gradually loses its memory about its previous state and so what we obtain are measures
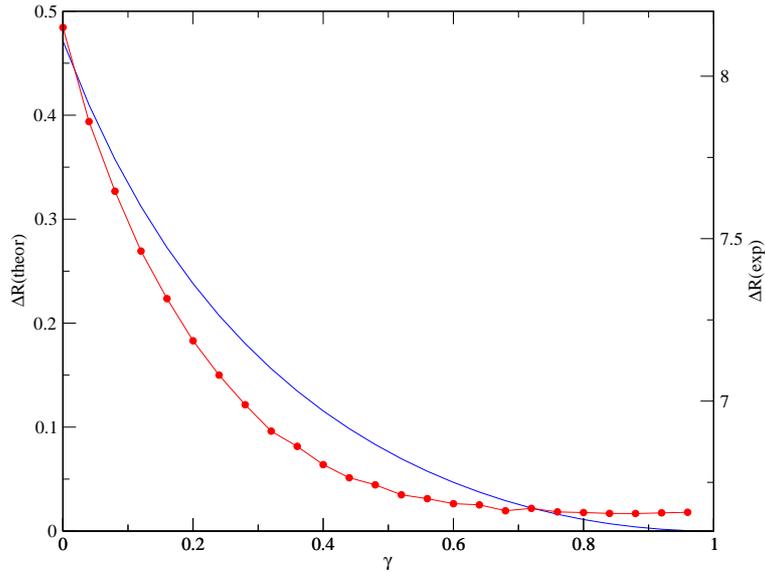
Figure 5. Plot of the theoretical curve for the inverse of the entropy per character and the numerical curve (circles) for the case when $\delta_t = 1$ step.

that can be considered as statistically independent, as we will not know for sure which will be the probability of finding the *source* in any of the two possible states. Now the dominant term will be the averaged probability –over all possible states– of finding the capital of the player with a value higher than the previous measure.

Considering the same stationary probabilities Π_0 and Π_1 given by Eq. 5 we can obtain the expressions for the average winning and losing probabilities over these two states as

$$
\begin{aligned}
p_{\text{win}} &= \sum_{j=1}^{M} \Pi_j \, p_j^i = \Pi_0 \, p_0' + \Pi_1 \, p_1' \\
p_{\text{lose}} &= \Pi_0 \cdot (1 - p_0') + \Pi_1 \cdot (1 - p_1').
\end{aligned}
\tag{7}
$$

It can be noticed in all Figs. 1,2,3,4 that as δ_t increases, the entropy curve acquires a more pronounced peak. This effect can be explained due to the fact that as the time between measures increases, the probability of finding the player with a higher (lower) capital increases (decreases). We can give an estimate of the probability that after δ_t rounds played, the capital of the player will be greater $(p_>)$ than it was initially by considering the probability that there have been less that $\delta_t/2$ loses in the δ_t coin tosses,

$$
p_> = \sum_{k=0}^{\frac{\delta}{2}} \binom{\delta_t}{k} \cdot p_{win}^{\delta_t - k} \cdot p_{lose}^k
\tag{8}
$$

These previous equations together with

$$
H = -p_> \cdot \log(p_>) - (1 - p_>) \cdot \log(1 - p_>),
\tag{9}
$$

allows one to obtain the entropy per character. In Fig. 6 we can see the curve for the variation of the inverse of information entropy for different values δ_t. In this case the negentropy decreases as δ_t increases, reflecting an increase of known information about the system.

This theoretical explanation also holds for the case of history dependent Parrondo's games and Parrondo's games with self–transition. For the latter case, though, we must take into account the fact that there exists a non negligible possibility that the capital can remain the same after a round played. In this case we need only
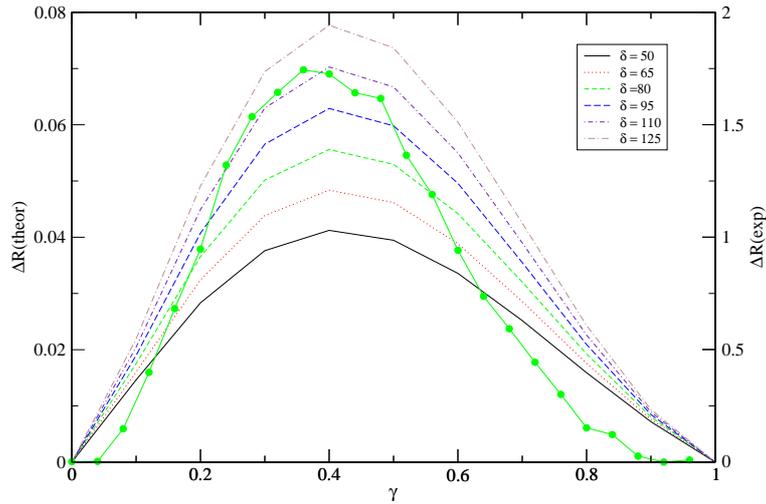
Figure 6. Plot of the theoretical curves for the inverse of the entropy per character for different values of δ_t, together with the numerical curve obtained for $\delta_t = 500$ steps.

to consider a source that will emit three different symbols, depending on whether the capital is greater, equal or smaller than our previous measure.

As a conclusion, we have demonstrated that there exists a transfer of information in the case of Parrondo's games, considered as a discrete–time and space version of the flashing ratchet. This effect takes place in every existing version of the games analyzed, showing the robustness of it. The qualitative analysis for the case of the original Parrondo's games have demonstrated the fact that the capital of the player can be thought of as a mixed source, composed of two ergodic sources, depending on whether the capital is a multiple of three or not.

We have also given some insight about the behavior of the information entropy for low and high values of δ_t, and why there is a decrease (increase) in the entropy per character h (information) as δ_t increases.

APPENDIX

We provide the C program used for creating the binary files that will be zipped for obtaining the entropy per character h.

```
int main()
{
    int i, j, k, ijk;
    int ijk1, ij, n;
    long nmax, irepet, icont, deltat, t;
    long  Nrepet, c, c0;
    double gamma, pa, pb1, pb2;
    double u, v, gain;
    unsigned char cero=0;
    unsigned char uno=255;
```

```
    srand48(342342);


    FILE *current;
    current = fopen("gain.dat","w");

/* We fix the lenght in bytes of the binary file   */

    Nrepet=300000;

/*  We impose the delay time between measurements  */

    deltat = 500;


/*  Probabilities of the original Parrondo's games */

    pa = 1./2.;
    pb1 = 1./10.;
    pb2 = 3./4.;


    ijk = 0;

    for (gamma = 0.; gamma <= 1.0; gamma = gamma + 0.04){

/*  we create a binary file for each value of gamma  */

    sprintf(q,"gamma%d.dat",ijk);
    FILE *data;
    data = fopen(q,"w");
    ++ijk;

    c0 = 0;
    c = 0;
    i= 0;
    ijk1 = 0 ;
    t = 0;

while (i<Nrepet){
    ++t;
        if (drand48() < gamma ){

/*      we  play game A     */

            if (drand48() < pa ){
            ++c;}
            else {
            --c;
            }
        ++ijk1;}
        else{
```

```
/*      we play game B   */

        if (fmod(c,3) == 0 ){
            if (drand48() < pb1 ){
            ++c;}
            else {
            --c;}
        }
        else {
            if (drand48() < pb2 ){
            ++c;}
            else {
            --c;}
        }
        ++ijk1;}

/*  every time that ijk1 equals deltat we evaluate the actual capital
of the player compared to its previous value    */

        if (ijk1 == deltat) {
        ijk1 = 0;
            if ((c-c0) > 0) {
            putc(cero,data);}
            else{
            putc(uno,data);
            }
        i++;
        c0 = c;
        }
        }

/* we also write the gain for every gamma value */

    gain =(double) c/((double) t);
    fprintf(current,"%f\t%f\n",gamma,gain);
    fclose(data);
    }
fclose(current);
return EXIT_SUCCESS;
}
```

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