

# Toward a thermo-hydrodynamic description of the Schrödinger equation via the Madelung formulation and the Fisher Information

Eyal Heifetz & Eliahu Cohen  
Tel Aviv University



## 1. From Schrodinger to Madelung

The non-relativistic time-dependent Schrödinger equation (TDSE) of a particle with mass  $m$ , under the presence of an external potential  $U$ :

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi = \left( \frac{\hat{\mathbf{p}}^2}{2m} + U \right) \Psi = \left( -\frac{\hbar^2}{2m} \nabla^2 + U \right) \Psi$$

with the wave function written in its polar form:

$$\Psi(\mathbf{r}, t) = \sqrt{\rho(\mathbf{r}, t)} e^{iS(\mathbf{r}, t)/\hbar}$$

together with the de Broglie guiding eq.:

$$\mathbf{u} = \frac{1}{m} \Re[\hat{\mathbf{p}}\Psi] = \nabla \left( \frac{S}{m} \right)$$

becomes the Madelung eq. (ME) and the continuity eq. of a pseudo-barotropic, irrotational, inviscid flow:

$$\frac{D}{Dt} \mathbf{u} = -\nabla \tilde{Q}(\rho) - \nabla \tilde{U} \quad \& \quad \frac{D}{Dt} \ln \rho = -\nabla \cdot \mathbf{u},$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \quad \text{is the material time derivative} \\ (\sim \text{represents a quantity per unit mass})$$

$$\tilde{Q} \equiv \frac{Q}{m} = -\frac{\hbar^2}{2m^2} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \quad \text{is the Bohm potential per unit mass.}$$

## 3. Classical inviscid, barotropic, potential, adiabatic fluid:

The Euler eq.  $\frac{D}{Dt} \mathbf{u} = -\frac{1}{\rho} \nabla P(\rho) - \nabla \tilde{U}$  &  $\mathbf{u} = \nabla \tilde{S}$

with the thermodynamics 1<sup>st</sup> law:  
(where  $I$  represents the internal energy)

$$\frac{D\tilde{I}}{Dt} = \frac{P}{\rho^2} \frac{D\rho}{Dt} \quad \Rightarrow \quad \frac{d\tilde{I}}{d\rho} = \frac{P}{\rho^2}$$

provides conservation of the kinetic, internal and potential energy:

$$E = \int \rho \tilde{E} dV = \langle \tilde{E} \rangle = \langle \tilde{K} + \tilde{I} + \tilde{U} \rangle$$

Energy however is not materially conserved by a single “fluid parcel” because the surrounding pressure,  $P$ , performs work to both accelerate and compress the fluid parcel.

The enthalpy:  $\tilde{E}_{nt} = \tilde{I} + P/\rho$  is related to  $Q$  by the Bernoulli potential:

$$\Rightarrow \nabla \tilde{Q}(\rho) = \frac{1}{\rho} \nabla P(\rho) = \nabla \left( \tilde{I} + \frac{P}{\rho} \right) \Rightarrow \tilde{Q} = \tilde{I} + \frac{P}{\rho} - \tilde{B}e(t)$$

The Bernoulli eq. takes a similar form to Hamilton-Jacobi **but** the enthalpy replaces the internal energy and the velocity potential  $S$  is the Action:

$$\frac{\partial \tilde{S}}{\partial t} = -(\tilde{K} + \tilde{Q} + \tilde{U}) = -\tilde{H} \quad \& \quad \frac{D\tilde{S}}{Dt} = \tilde{K} - (\tilde{Q} + \tilde{U}) = \tilde{L}$$

$$\text{if: } \int P dv = \tilde{B}e(t) \quad \Rightarrow \quad \langle \tilde{Q} \rangle = \langle \tilde{I} \rangle \quad \& \quad \langle \tilde{E} \rangle = \langle \tilde{H} \rangle$$

One conclusion of this approach, is that incompressibility, i.e.  $\nabla \cdot \mathbf{u} = 0$

is equivalent to the non-spreading property. Furthermore,  $\frac{\partial \mathbf{u}}{\partial t} = -\nabla(\tilde{K} + \tilde{Q} + \tilde{U})$

In 1D this implies that  $\tilde{Q} + \tilde{U} = a(t)x + b(t)$  which is indeed the case of the Airy wave packet, the quantum bouncer and the harmonic oscillator.

## 2. Fisher information and quantum fluid thermal (internal) energy

The Schrödinger equation conserves the total energy integral:

$$\langle \tilde{E} \rangle = \frac{1}{m} \int \Psi^* \left( -\frac{\hbar^2}{2m} \nabla^2 + U \right) \Psi dV = \langle \tilde{K} + \tilde{Q} + \tilde{U} \rangle, \quad \tilde{K} = \mathbf{u}^2/2$$

$Q$  is not positive definite but after integration by parts:

$$\langle \tilde{Q} \rangle = \left\langle \frac{1}{2} \left[ \frac{\hbar}{2m} \nabla(\ln \rho) \right]^2 \right\rangle = \frac{\hbar}{8m^2} FI,$$

where the Fisher information is:  $FI \equiv \int \rho [\nabla(\ln \rho)]^2 dV$

We note that if we define:  $\hat{\mathbf{p}}\Psi = [-i\hbar \nabla \ln \Psi]\Psi \equiv \mathbf{p}\Psi,$

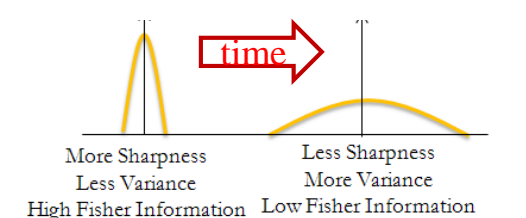
$$\mathbf{v} \equiv \frac{\mathbf{p}}{m} = \mathbf{v}_r + i\mathbf{v}_i \quad \Rightarrow \quad \mathbf{v}_r = \mathbf{u} = \nabla \left( \frac{S}{m} \right), \quad \mathbf{v}_i = -\frac{\hbar}{2m} \nabla(\ln \rho),$$

$\mathbf{v}_i$  is proportional to the Fisher score, having a zero expectation value, as is required from random thermal motion, hence defining the internal energy  $I$ :

$$\tilde{I} = \mathbf{v}_i^2/2 \quad \Rightarrow \quad \langle \tilde{Q} \rangle = \langle \tilde{I} \rangle \quad \Rightarrow \quad \langle \tilde{K} + \tilde{Q} \rangle = \frac{1}{2} \langle \mathbf{v}_r^2 + \mathbf{v}_i^2 \rangle,$$

can be regarded as the total kinetic (hydrodynamic macroscopic plus thermal microscopic) energy of the flow representing the quantum particle.

Adiabatic expansion of this “quantum gas” tends to decrease the thermal internal energy ( $FI$ ) with time and can serve as an alternative measure of negative entropy.



## 4. The strange thermo-hydrodynamics of the “quantum gas”:

The TDSE yields:  $\frac{\partial \tilde{S}}{\partial t} = -(\tilde{K} + \tilde{Q} + \tilde{U})$

hence, here the Bohm potential  $Q$  is the starting point for the thermodynamics.

$$\frac{1}{\rho} \nabla P = \nabla \tilde{Q}(\rho) \quad \Rightarrow \quad P = -\Pi(\rho) + \phi(t) \quad \Pi(\rho) = \left( \frac{\hbar}{2m} \right)^2 \rho \nabla^2 \ln \rho$$

On the other hand, for  $\tilde{I} = \mathbf{v}_i^2/2$  the 1<sup>st</sup> law  $\frac{d\tilde{I}}{d\rho} = \frac{P}{\rho^2} \Rightarrow P = \Pi$

Furthermore  $\tilde{Q} = -\tilde{I} + \frac{\Pi}{\rho}$  but because  $\int \Pi dv = 2 \langle \tilde{Q} \rangle \Rightarrow$

in the domain average sense:  $\langle \tilde{Q} \rangle = \langle \tilde{I} \rangle \quad \& \quad \langle \tilde{E} \rangle = \langle \tilde{H} \rangle$

## Conclusions:

- The above analysis suggests a novel equation of state for the quantum gas that emphasizes its uniqueness.
- Momentum can be understood as having an imaginary part responsible for the internal thermal energy/Bohm potential, both related to the Fisher information.
- The proposed description may provide a more intuitive understanding of quantum systems, e.g. the non-spreading property.
- Future work will confront this seemingly local description of quantum theory with the apparent nonlocality of quantum mechanics. The link is the instability of the above nonlinear equations.
- More details in : <http://arxiv.org/pdf/1501.00944v1.pdf> and subsequent publications.