Thermodynamics of trajectories of a harmonic oscillator…

André Xuereb (University of Malta & Queen's University Belfast)

Simon Pigeon, Lorenzo Fusco, Gabriele De Chiara & Mauro Paternostro (Queen's University Belfast)

Reference: S. Pigeon, *et al.*, arXiv:1411.2637 (2014)

Thanks to

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- Thanks also to Igor Lesanovsky and Juan P. Garrahan from the University of Nottingham

One more thing

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Meetings in Malta!

- Mid-November 2015: ESR Workshop hosted by MP1403
- February 21 to 24 (tbc): 6th Working Group Meeting of MP1209

http://qutmalta.sciencesconf.org/

Thermodynamics of trajectories

- Consider a thermodynamic quantity with a control parameter λ
- We can associate a corresponding free energy function $\theta(\lambda)$
- Non-analyticities in $\theta(\lambda)$ correspond to phase transition points
- In dynamical systems, one can analyse order parameters in a similar manner

Counting processes

- We define a counting process K, e.g., the net number of excitations emitted
- Given the density matrix ρ, project onto each K-subspace

$$\rho_K(t) \coloneqq \Pi_K \rho(t) \Pi_K$$

- From this, construct the "s-biased ensemble"

$$\rho_s(t) \coloneqq \sum_K e^{-sK} \rho_K(t)$$

Counting processes

- The partition function associated with ρ_s takes a large-deviation form

$$Z(s,t) \coloneqq \mathrm{Tr}\{\rho_s(t)\} \xrightarrow[t\to\infty]{} e^{t\theta(s)}$$

- $\rho_s(t)$ obeys a modified (trace non-preserving) master equation

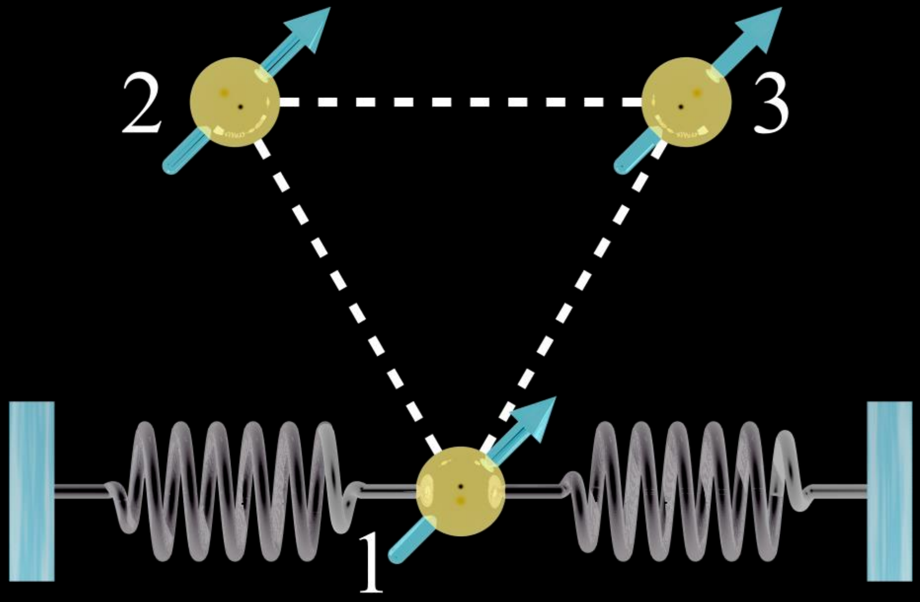
Counting processes

- $\theta(s)$ is the free-energy corresponding to K
- It gives access to the statistics of K, e.g.,

$$\langle K\rangle/t = -\partial_s\theta(s)\Big|_{s=0}$$

- Non-analyticities in $\theta(s)$ correspond to phase transitions **in the dynamics**

Three-spin system



Three-spin system

- We consider three spins-1/2 arranged in a triangular configuration
- Two spins are pinned, whereas the third is allowed to move
- The basic model:

$$\hat{H}_{\mathrm{m}} = \alpha \sum_{i} \sigma_x^{(i)} + \sum_{\langle i,j \rangle} \sigma_x^{(i)} \sigma_x^{(j)} - B \sum_{i} \sigma_z^{(i)}$$

- The motion of spin 1 modulates the second term, yielding

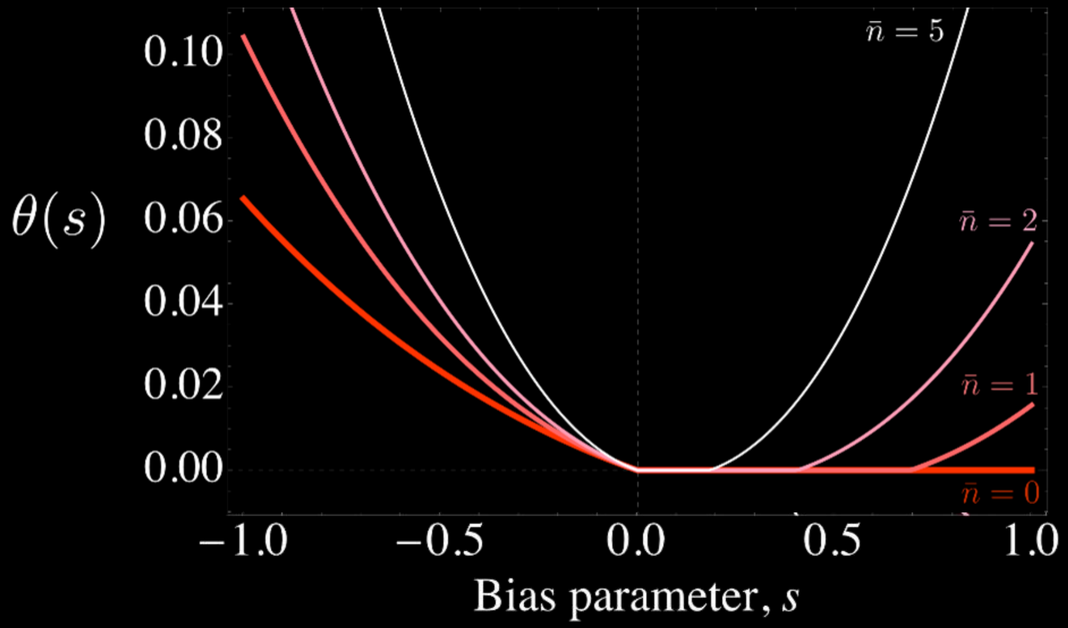
$$\hat{H}_{\mathrm{s-m}} = g\, \hat{x}\, \sigma_x^{(1)} \left(\sigma_x^{(2)} - \sigma_x^{(3)} \right)$$

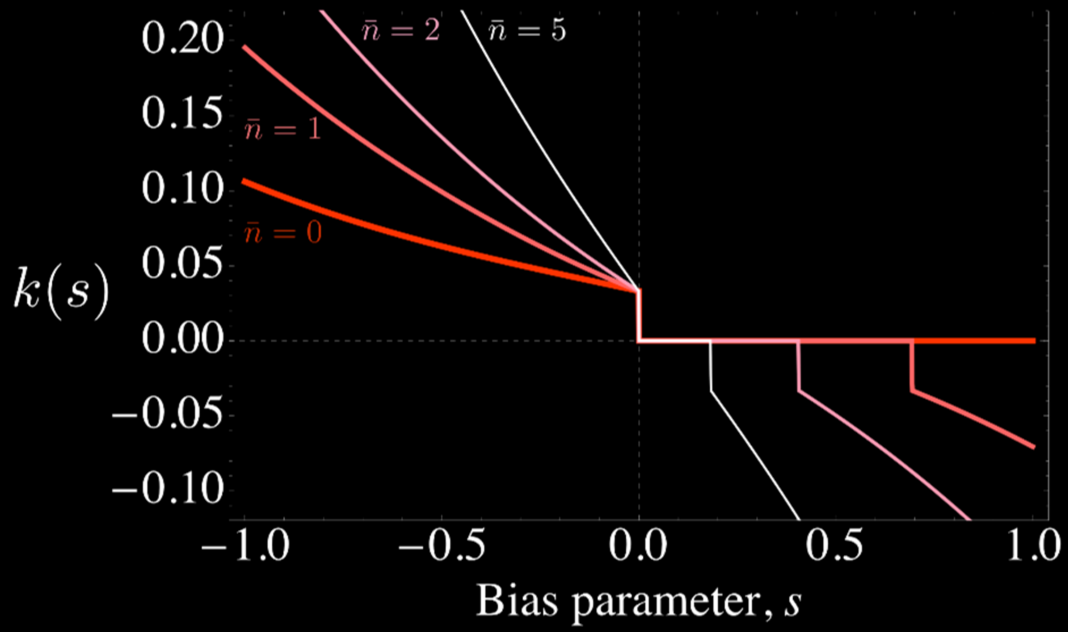
- We open the system by damping $\hat{x}$

Three-spin system

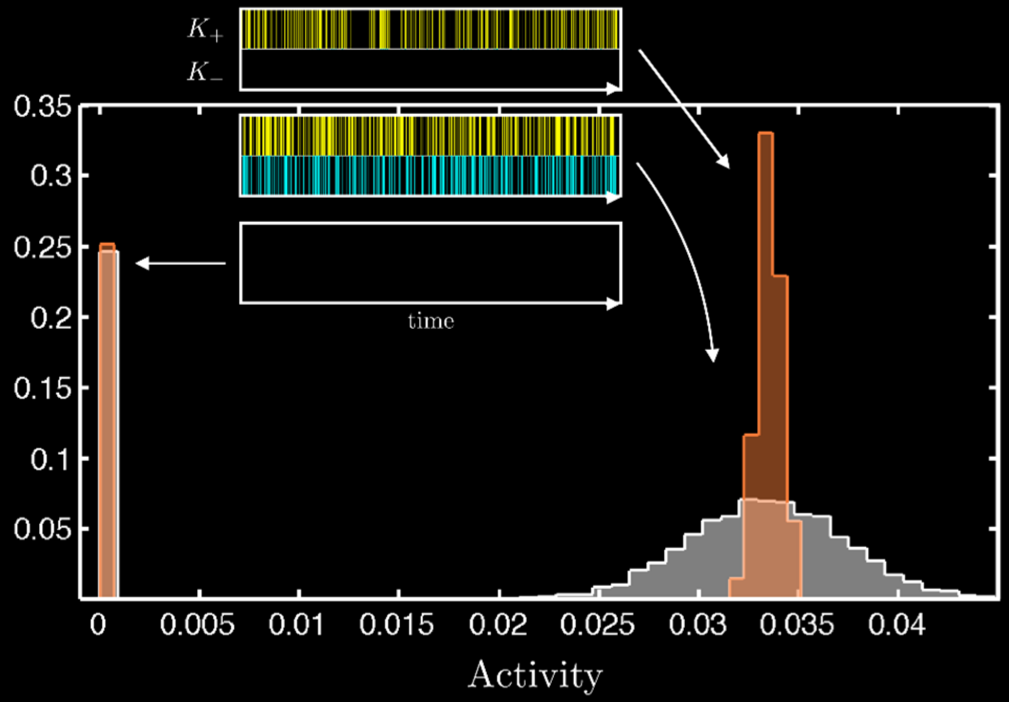
- Following usual quantum-optics methods, we eliminate the motion
- This yields an effective damping of the spin system through the operator
$$\sigma_-^{(1)}\left(\sigma_-^{(2)} - \sigma_-^{(3)}\right)$$
- This collective dissipation partitions the Hilbert space into two:
  - An "active" phase which emits excitations to the bath
  - An "inactive" phase that emits no excitations
- Dissipation of the individual spins also plays a role, as we shall see

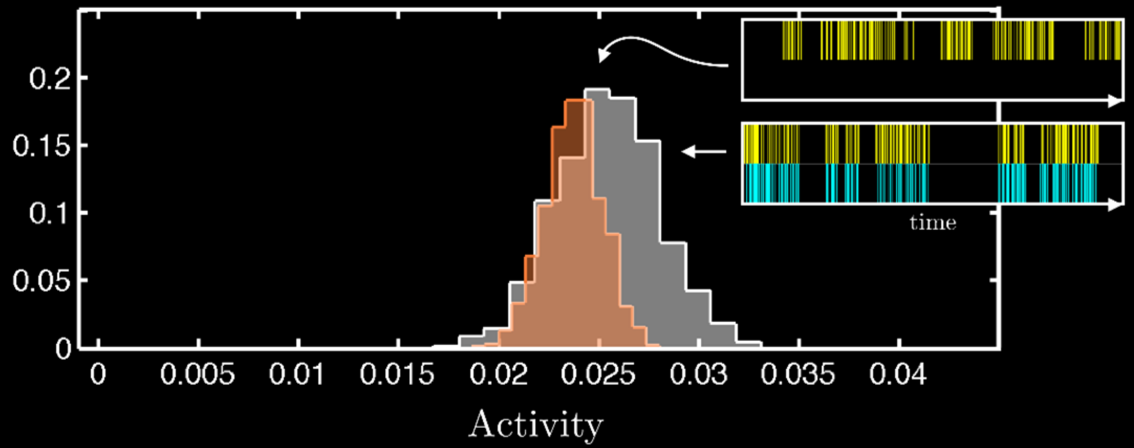
Three-spin system





Three-spin system





Evaluating $\theta(s)$

- Consider a master equation in Lindblad form
$$\dot{\rho} = -i\left[\hat{H}, \rho\right] + \mathcal{L}[\rho]$$
  where
$$\mathcal{L}[\rho] = \gamma(\bar{n}+1)\left(\hat{a}\rho\hat{a}^\dagger - \tfrac{1}{2}\{\rho, \hat{a}^\dagger\hat{a}\}\right) + \gamma\bar{n}\left(\hat{a}^\dagger\rho\hat{a} - \tfrac{1}{2}\{\rho, \hat{a}\hat{a}^\dagger\}\right)$$
- This represents a system that exchanges energy with a bath:
  - To the bath with a rate $\gamma(\bar{n}+1)$
  - From the bath with a rate $\gamma\bar{n}$

Evaluating $\theta(s)$

- Let us now count quanta entering or leaving the system along this channel
- We can obtain ρ_s directly by using the modified master equation

$$\dot{\rho}_s = -i\left[\hat{H}, \rho_s\right] + \mathcal{L}[\rho_s] + \mathcal{L}_s[\rho_s] = \mathcal{W}_s[\rho_s]$$

where

$$\mathcal{L}_s[\rho_s] = \gamma(\bar{n}+1)(e^{-s}-1)\hat{a}\rho\hat{a}^\dagger + \gamma\bar{n}(e^{s}-1)\hat{a}^\dagger\rho\hat{a}$$

- The Liouvillian for any other baths is not modified
- $\theta(s)$ is simply the eigenvalue of $\mathcal{W}_s[\cdot]$ with largest real part

Evaluating $\theta(s)$

- For a simple three-spin system, we need to diagonalise a 64×64 matrix
- What about continuous-variable systems?

Gaussian systems

- Gaussian states comprise a wide variety of states
- Defined by having a Gaussian Wigner quasiprobability
- Gaussianity is conserved under the action of a quadratic Hamiltonian

Gaussian systems

- Wigner quasiprobability distribution is obtained via a “Fourier transform” of the density operator

$$W(x,p,t) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} e^{2ipy/\hbar} \langle x - y|\rho(t)|x + y\rangle \, \mathrm{d}y$$

- Gaussian states have a Gaussian Wigner function

$$W(x,p,t) = A(t)\exp[-\boldsymbol{r} \cdot \sigma(t)^{-1} \cdot \boldsymbol{r}]$$

- $\boldsymbol{r} = (x,p)^{\mathrm{T}}$ and $\sigma(t)$ is the **covariance matrix**

Gaussian systems

- Usually, the normalising factor $A(t)$ is such that

$$\iint_{-\infty}^{\infty} W(x,p,t)\,\mathrm{d}x\,\mathrm{d}p = \mathrm{Tr}\{\rho\} = 1$$

- Our modified master equation is not trace-preserving

$$\mathrm{Tr}\{\rho_s\} \neq 1$$

- Indeed (as $t \to \infty$),

$$\theta(s) = \frac{1}{t}\ln Z(s,t) = \frac{1}{t}\ln \mathrm{Tr}\{\rho_s(t)\} = \frac{1}{t}\ln A(t)$$

Gaussian systems

- It turns out that all the information about $\theta(s)$ is contained within $A(t)$
- Consider a thermal state, where $\sigma(t) = \begin{pmatrix} v(t) & 0 \\ 0 & v(t) \end{pmatrix}$:

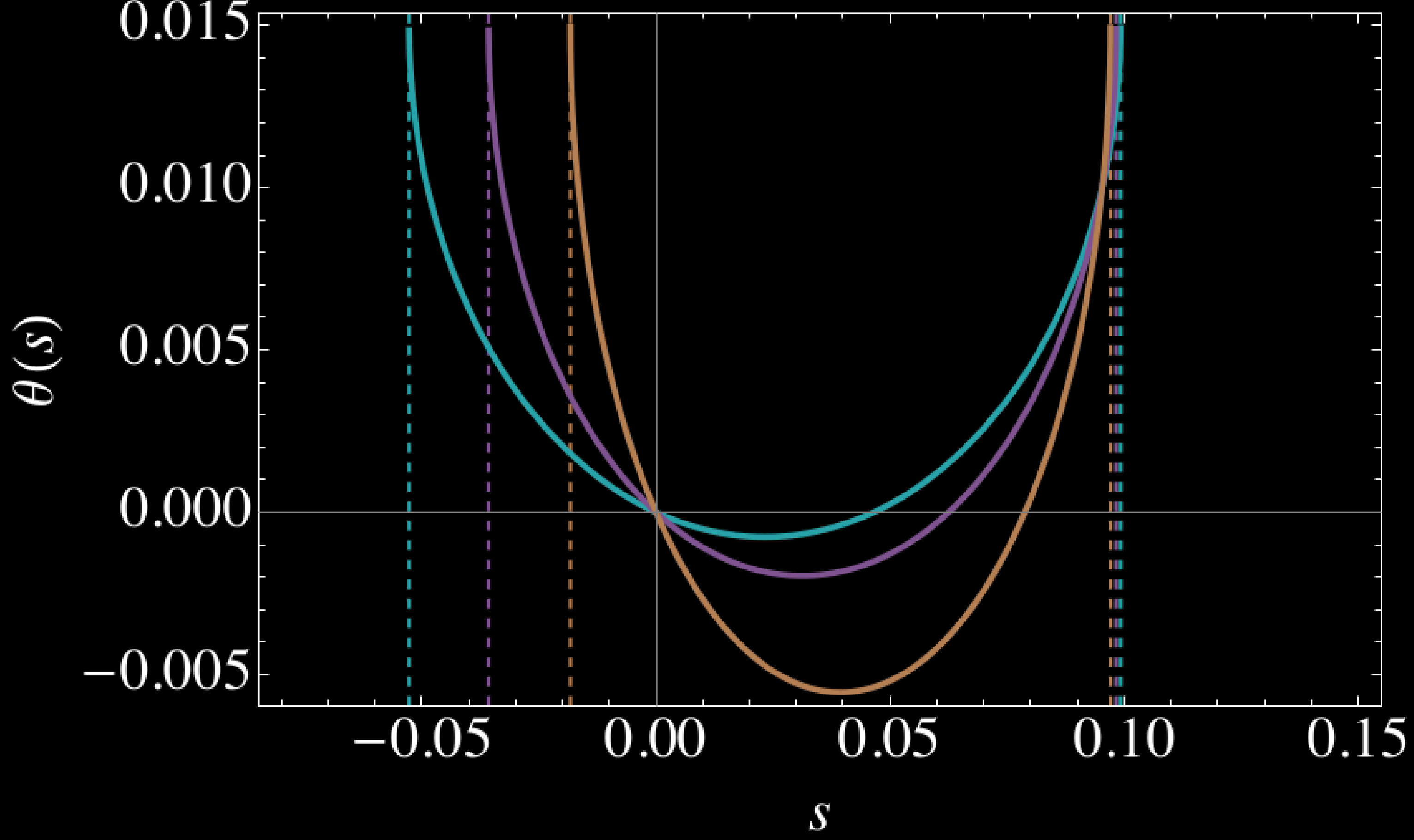
$$\theta(s) = 2 \lim_{t\to\infty} \frac{1}{t} \int^{t} [f_+(s)v(\tau) - f_-(s)] \, \mathrm{d}\tau$$

- The functions $f_\pm(s)$ are simple functions of the system parameters
- In many cases, $v(t)$ reaches a steady state and the above equation can be evaluated in a straightforward manner

Gaussian systems

- We consider a generic system of a harmonic oscillator coupled to N baths
- Define a counting process that counts the excitations entering or leaving the system through one particular bath
- For this system, $\theta(s)$ can be found analytically

Gaussian systems



Gaussian systems

- The dynamics is invariant under the replacement $s \to s_0 - s$, where s_0 depends on the bath temperature
- Thos property is known as a Gallavotti–Cohen symmetry [1]
- Using this, one can show that the system obeys a fluctuation relation

$$\frac{p_K}{p_{-K}} = e^{s_0 K}$$

- Here $p_K = \lim_{t\to\infty} \mathrm{Tr}\{\rho_K\}$ is the infinite-time probability of counting K events

[1] J. L. Lebowitz, H. Spohn, J. Stat. Phys. **95**, 333 (1999).

Gaussian systems

- The end points are related to the tails of the probability distribution, which are exponential
- Visually similar results were obtained for *classical* harmonic oscillators [2]

[2] H. C. Fogedby, and A. Imparato, J. Stat. Mech. **2011**, P05015 (2011).

Gaussian systems

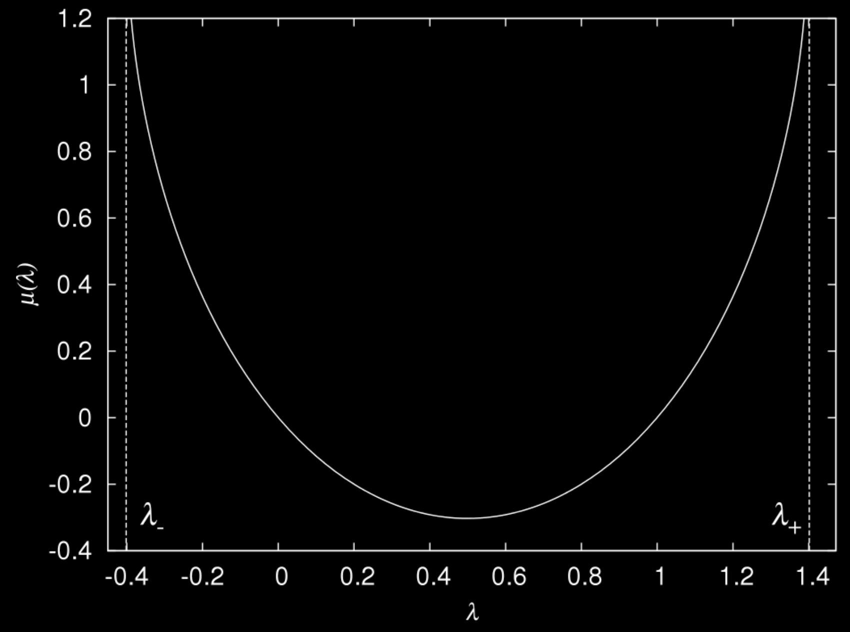


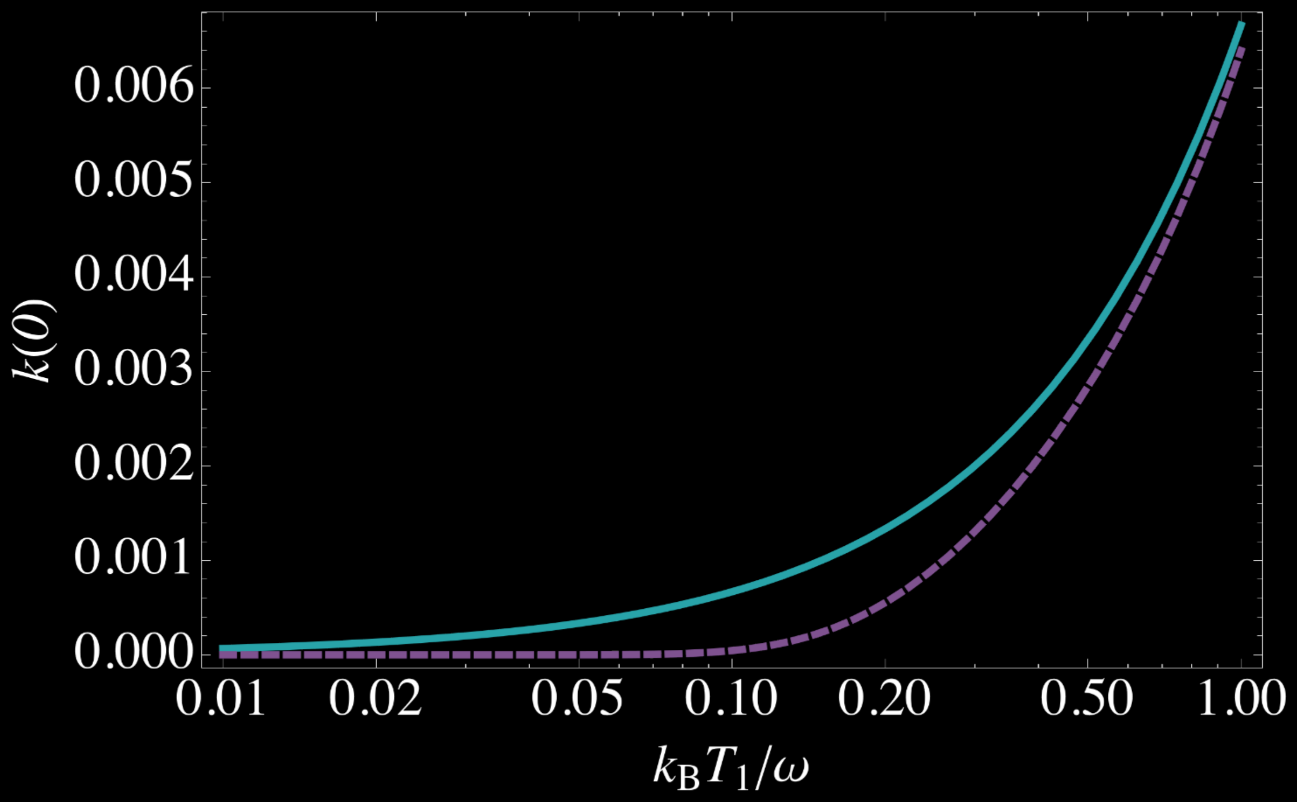
Figure 2. Large deviation function $\mu(\lambda)$ as a function of λ, as given by equation (4.1), for $\Gamma_1 = 1$, $\Gamma_2 = 2$, $T_1 = 1$, $T_2 = 2$. The shape is that of a half circle lying between the branch points $\lambda_{\pm}$, as given by (4.2).

[2] H. C. Fogedby, and A. Imparato, J. Stat. Mech. **2011**, P05015 (2011).

Gaussian systems

- Comparison between the two shows that:
  - For high temperatures, the two results agree perfectly
  - For low temperatures, the quantum version is **less active**; there is a suppression of the mean net exchange of excitations to/from the bath

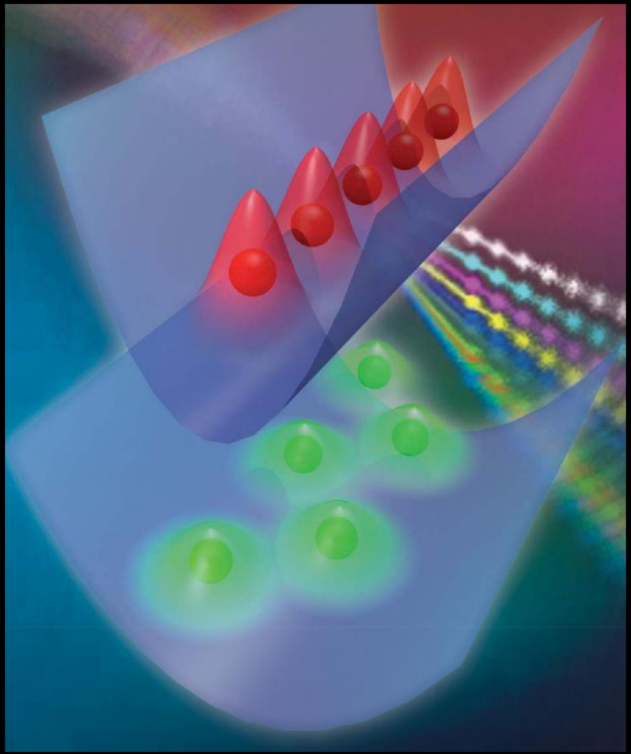
Gaussian systems



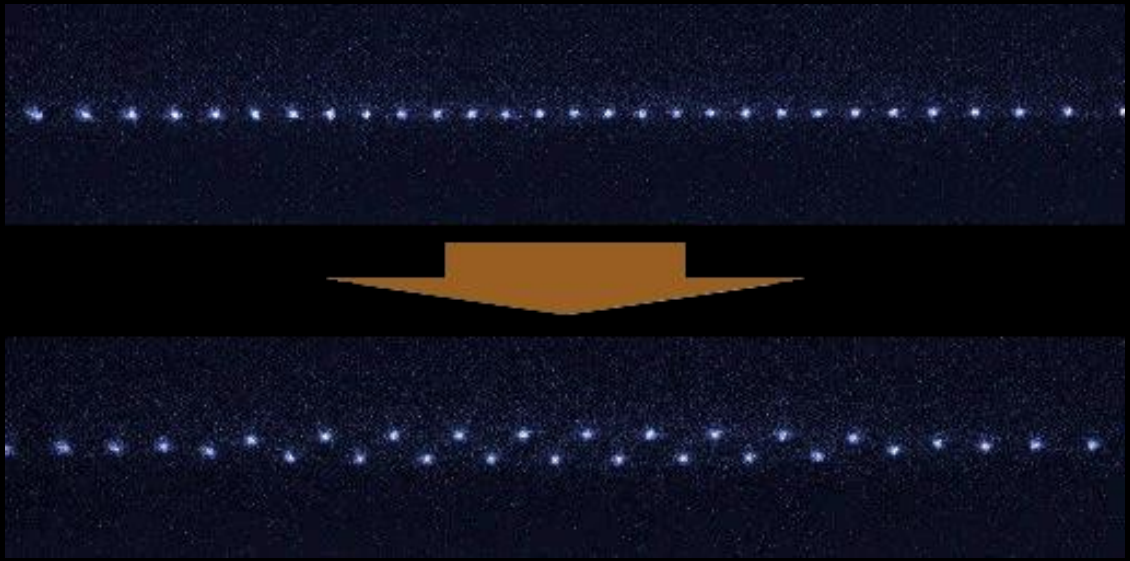
Gaussian systems: Outlook

- Not shown here: Driving the system can be incorporated
- We are looking at applying this method to handle any system of M oscillators coupled to N baths
- Analytical results are hard, but numerics are easy (solving an algebraic Riccati equation)
- Systems of interest include the linear to zig-zag transition in chains of trapped ions

Gaussian systems: Outlook



[3]



[4]

[3] Cover image from Ann. Phys. **10–11** (2013).

[4] A. del Campo, T. W. B. Kibble, and W. H. Zurek, J. Phys. **25**, 404210 (2013).

Conclusions

- Thermodynamics of trajectories can reveal information about the dynamics of an open quantum system
- For continuous variable systems, the problem is a difficult one
- Restricting ourselves to Gaussian systems, $\theta(s)$ may be found analytically
- In the high-temperature limit, our results perfectly match classical ones
- For very low temperatures, we find that quantum systems are *quieter*
- We are working towards analysing networks of oscillators from this point of view

Thank you!

Thank you for your attention.

References:

- *Oscillator and N baths:* S. Pigeon, *et al.*, arXiv:1411.2637 (2014)
- *Three-spin system:* S. Pigeon, *et al.*, New J. Phys. **17**, 015010 (2015)

Background:

- J. P. Garrahan, and I. Lesanovsky, Phys. Rev. Lett. **104**, 160601 (2010)